Jurnal Pendidikan Edutama
Volumes 11 Number 2 July 2024
P-ISSN: 2339-2258 | E-ISSN: 2548-821X
IKIP PGRI Bojonegoro

Exploring the Correlation of Recognizing Proof Techniques and Definitions (Theorems) on the Ability of Indonesian Students in Mathematical Proof

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Keywords

Recognizing Proof Teaching, Mathematical Proof, Correlation

Abstract

The aim of this research is to measure the extent of the correlation between the ability to recognize proof techniques and the ability to recognize definitions/theorems with the ability to prove in the Real Analysis course. This study tests the associative hypothesis of two independent variables (the ability to recognize proof techniques and definitions/theorems) with one dependent variable (the ability in mathematical prove). The research sample consists of a total of 60 students currently enrolled in the real analysis course. The study was lecture-based, involving tasks such as identifying proof techniques in provided proofs from textbooks, writing definitions/theorems in proofs, and conducting proof tests to measure proof abilities. Subsequently, the data was analyzed using a multiple correlation test between the two independent variables and one dependent variable with the assistance of SPSS version 25. The correlation coefficient obtained is 0.821, indicating a very strong relationship between the ability to recognize proof techniques and the skill of recognizing definitions/theorems together with the ability to prove.

Introduction

The real analysis course is one of the compulsory courses for mathematics education students in Indonesia. Real analysis serves as the foundation for learning design research, particularly in the form of proofs (Minggi et al., 2021), especially (Helma et al., 2018) (CadwalladerOlsker, 2011). Real analysis courses include definitions and theorems. Real analysis is one of the feared courses due to its difficulty level. Real analysis material starts from definitions that do not provide motivation and leads into formal theorem understanding, which is not easily grasped (Bressoud, 2020). Definitions or theorems underlying understanding influence the proofs of subsequent theorems (Minggi et al., 2021). Definitions and theorems contain logically interconnected structures that are needed in the proof process of subsequent theorems (A. Selden et al., 2018). Each statement originates from previous statements or other concepts/knowledge, influencing the understanding and solving of mathematical problems (Alcock & Weber, 2005). The issue of solving mathematical problems involves not only analytical solutions but also mathematical proof solving. Mathematical proof solving requires high cognitive abilities. High cognitive abilities involve analysis skills, reasoning abilities leading to deductive reasoning (logical deduction).

Previous research has consistently shown that proof-solving ability is a frequently and importantly studied aspect in mathematics education. Several mathematicians consider proof as the most important and fundamental skill in mathematics (Alcock & Weber, 2005) (Feriyanto, 2018) (Yesilyurt Cetin & Dikici, 2023) (Güler & Dikici, 2014) (Güler, 2014) (Güler, 2016) (CadwalladerOlsker, 2011), and it continues to be studied to this day (Arana & Stafford, 2023). Mathematical proof-solving ability requires proof skills (Sommerhoff et al., 2021). However, there are still problems encountered by students in proof-solving (Helma et al., 2018), and other difficulties faced by university students regarding proving (Stylianou, Blanton, & Rotou, 2015), including difficulties in recognizing definitions/theorems (Dawson, 2006) (Minggi et al., 2021) (Morali & Filiz, 2023). Most students still believe that proof activities are not necessary in real life (Coe & Ruthven, 1994). When students are faced with a theorem to be proved, they often do not know where to start the proof (Selden & Selden, 2016) (Selden et al., 2018), and a proof schema is needed (Junarti et al., 2019). Other difficulties arise due to the complexity of the problems encountered in the proof process (Arana & Stafford, 2023), and epistemological difficulties where students are unable to use concepts related to previous definitions/theorems (Isnani et al., 2022) (Minggi et al., 2021). The inability of students as mentioned above becomes an obstacle in performing proofs.

The inability of students to recognize or understand definitions/theorems is not the only obstacle; certainly, there are many other factors that hinder proof-solving, such as the inability to recognize proof techniques. In proof-solving, constructing the proof/framework is necessary (Benkhalti et al., 2017) (Nadlifah, 2020) (Selden et al., 2016) (Selden & Selden, 2016). This framework requires proof techniques (Celluci, 2008) (Hsu & Jos, 2008) (Minggi et al., 2021)(Rocha, 2019) (Stefanowicz et al., 2014). Knowledge of proof techniques is important for students to choose the types of proof techniques they will use in the proof process and facilitate starting the proof.

The success of students in mathematics is determined by their belief in the importance or correlation with the topics they have to learn (Schoenfeld, 1989; Stylianou, Blanton, & Rotou, 2015). This can arise from the problems students encounter in proof-solving or because they perceive proof as unnecessary in learning mathematics (Furinghetti, Olivero, & Paola, 2001). Surprisingly, proof-solving skills can lead to synergistic integration as a resource during proof construction and can be beneficial for students (Sommerhoff et al., 2021). In Selden's study, it is quite common to find students not using definitions correctly (A. Selden et al., 2018). Additionally, the ability to choose proof techniques also determines the steps in proof-solving (Minggi et al., 2021). The sequence of concepts, definitions, and previous theorems is part of the learning trajectory involved in proof-solving (Minggi et al., 2021). Mathematical proof requires deductive reasoning skills. Deductive reasoning is a complex concept that requires sequential steps (Wang et al., 2020). Based on the above studies, it is evident that there are many factors strongly correlated with proof-solving ability. Therefore, the aim of this research is to measure the extent of the relationship between the ability to recognize proof techniques and the skill of recognizing definitions/theorems with proof-solving ability in the real analysis course.

Method

This study is a quantitative research with interval data type. It tests an associative hypothesis based on the theory that proof-solving ability is related to the ability to recognize proof techniques and the skill of recognizing definitions/theorems. Therefore, the study tests the hypothesis of the association between two independent variables (x_1 =ability to recognize proof techniques and x_2=ability to recognize definitions/theorems) and one dependent variable (y = ability in mathematical prove). The research design is lecture-based and task-based, so the instruments were not tested for validity. The study was conducted on first-year students in the second semester of the mathematics education program in Indonesia, with a total sample of 60 students (using total sampling) enrolled in the real analysis course. The real analysis material studied was limited to the sub-chapter "algebra in real numbers and properties of order in R (covering 4 definitions and 12 theorems)".

All samples were introduced to various proof techniques along with examples during the first meeting. At the end of the first meeting, the samples were tasked with identifying/categorizing proof techniques used in 19 proofs of the theorems provided in the Real Analysis book by Bartle. Subsequently, in the second, third, and fourth meetings, learning was conducted by introducing and explaining definitions and theorems, as well as methods of proving them. In the last meeting, the samples were given a test on proof-solving ability through three proof-solving questions (related to the material covered).

The first independent variable, x_1 (the ability to recognize proof techniques), was measured through a task of categorizing proof techniques in 23 proofs. Each sample was given a score of 1 if categorized correctly (according to the category) and a score of 0 if categorized incorrectly (not according to the category), with a final score on a scale of 100.

The second independent variable, x_2 (the ability to recognize definitions/theorems), was measured through a task where students were asked to write down the definitions/theorems related to proving 12 theorems from the book (from the material being taught). They were given a score of 1 if categorized correctly (or according to the related definition/theorem) and a score of 0 if categorized incorrectly (not according to the related definition/theorem), with a scoring scale of 100.

To measure the dependent variable, y (proof-solving ability), a test consisting of 3 proof-solving questions was administered, with a scoring scale of 100.

Data analysis was conducted using basic statistics, namely the product moment formula and multiple correlation using SPSS version 25. This calculation was used to measure: 1) the extent of the relationship between variables x_1 and x_2, 2) the extent of the relationship between variable x_1 and variable y, 3) the extent of the relationship between variable x_2 and variable y, and 4) the extent of the relationship between the independent variables x_1 and x_2 simultaneously with the dependent variable y. The strength of the relationship of each variable can be used to recommend the importance of these independent variables in improving mathematical proof-solving skills.

Results and Discussion

Based on the analysis results of the tasks and tests used to measure the abilities of the three variables x_1 and x_2, and y using SPSS version 25, the following results are obtained as presented in Table 1 Descriptive Statistics below.

Table 1 Descriptive Statistics

	N	Mean	Std. Deviation	Minimum	Maximum
x_1	60	65.90	7.290	52	83
x_2	60	60.37	7.066	50	75
y	60	68.27	9.544	20	85

In Table 1 above, the mean value for the first independent variable group x_1, which is the ability to recognize proof techniques, is 65.90, with a maximum value of 83 and a minimum value of 52. This indicates that students are fairly capable of recognizing types/forms of proof techniques.

Meanwhile, the mean for the second independent variable group x_2, which is the ability to recognize definitions/theorems in proofs, is 60.37, with a maximum value of 75 and a minimum value of 50. These results are below the mean for the ability to recognize proof techniques, indicating that students may have difficulty linking definitions/theorems to proofs.

Furthermore, for variable y, the mean value is 68.27, representing the mean of students' proof-solving abilities, with a highest score of 85 and a lowest score of 20. This indicates that, on average, proof-solving ability is above the mean of both the ability to recognize proof techniques and the ability to recognize definitions/theorems. However, based on the maximum values, it is above the maximum values of both variables x_1 and x_2.

On the other hand, based on the minimum values, it is below the minimum values of variables x_1 and x_2. A tentative conclusion suggests that proof-solving ability is above the other two abilities (the ability to recognize proof techniques and the ability to recognize definitions/theorems).

3.1. Simple Correlation Test (Variables x_1 and y)

In the simple correlation test between variable x_1 and variable y (which is the correlation between the ability to recognize proof techniques and proof-solving ability) using SPSS version 25 analysis, the results are presented in Table 2 below.

Table 2 Correlations between x_1 and y

		x_1	y
x_1	Pearson Correlation	1	.821**
	Sig. (2-tailed)		.000
	Sum of Squares and Cross-products	3135.400	3370.600
	Covariance	53.142	57.129
	N	60	60
y	Pearson Correlation	.821**	1
	Sig. (2-tailed)	.000	
	Sum of Squares and Cross-products	3370.600	5373.733
	Covariance	57.129	91.080
	N	60	60

In Table 2 above, the correlation coefficient between variable x_1 and variable y is 0.821. This value indicates a very strong categorized level of relationship.

3.2. Simple Correlation Test (Variable x_2 and y)

In the simple correlation test between variable x_2 and variable y (which is the correlation between the ability to recognize definitions/theorems and proof-solving ability) using SPSS version 25 analysis, the results are presented in Table 3 below.

Table 3 Correlations between x_2 and y

		x_2	y
x_2	Pearson Correlation	1	.720**
	Sig. (2-tailed)		.000
	Sum of Squares and Cross-products	2945.933	2866.133
	Covariance	49.931	48.579
	N	60	60
y	Pearson Correlation	.720**	1
	Sig. (2-tailed)	.000	
	Sum of Squares and Cross-products	2866.133	5373.733
	Covariance	48.579	91.080
	N	60	60

In Table 3 above, the correlation coefficient between variable x_2 and variable y is 0.720. This value indicates a strong categorized level of relationship.

3.3. Simple Correlation Test (Variable x_1 and x_2)

In the simple correlation test between variable $x_1$and variable x_2 (which is the correlation between the ability to recognize proof techniques and proof-solving ability) using SPSS version 25 analysis, the results are presented in Table 4 below.

Table 4 Correlations between x_1 and x_2

		x_1	x_2
x_1	Pearson Correlation	1	.868**
	Sig. (2-tailed)		.000
	Sum of Squares and Cross-products	3135.400	2638.200
	Covariance	53.142	44.715
	N	60	60
x_2	Pearson Correlation	.868**	1
	Sig. (2-tailed)	.000	
	Sum of Squares and Cross-products	2638.200	2945.933
	Covariance	44.715	49.931
	N	60	60

In Table 4 above, the correlation coefficient between variables x_1 and x_2 is 0.868. This value indicates a very strong categorized level of relationship.

3.4. Multiple Correlation Test (Variable x_1, x_2 and y)

In the multiple correlation test between variables x_1 and x_2 simultaneously with variable y (which is the correlation between the ability to recognize proof techniques and the ability to recognize definitions/theorems with proof-solving ability) using SPSS version 25 analysis, the results are presented in Table 5 below.

Table 5 Model Summary

				Std. Error	Change Statistics				
Model	R	R Square	Adjusted R Square	of the Estimate	R Square Change	F Change	df1	df2	Sig. F Change
1	.821[a]	.675	.663	5.539	.675	59.063	2	57	.000

a. Predictors: (Constant), X2, X1

In Table 5 above, the correlation coefficient between variables x_1 and x_2 with variable y is 0.821. This value indicates a very strong categorized level of relationship.

The relationship scheme between the two independent variables and one dependent variable is shown in Figure 1 below.

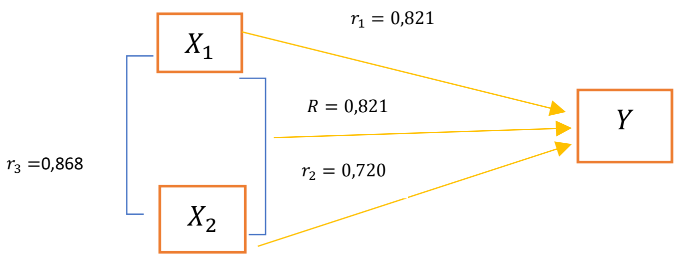

Figure 1 The correlation between independent variables, between independent variables and the dependent variable, and the multiple correlation between two independent variables and one dependent variable.

In Figure 1, there are 4 relationships: 1) r_1 is the correlation coefficient between variable x_1 and y; 2) r_2 is the correlation coefficient between variable x_2 and y; 3) r_3 is the correlation coefficient between variable x_1 and x_2, and 4) the multiple correlation between variablesx_1 and x_2 with variable y is denoted by R_{y,x_1,x_2}. Based on the respective correlation coefficients in Figure 1 above, $r_1 = r_{x_1y} = 0.821$, $r_2 = r_{x_2y} = 0.720$, $r_3 = r_{x_1x_2} = 0.868$, which means that the relationship between the two variables is quite strong. Meanwhile, the multiple correlation between the ability to recognize proof techniques and the ability to recognize definitions/theorems together with proof-solving ability is obtained as $R = 0.821$. Based on the magnitude of the multiple correlation coefficient compared to the individual correlation coefficients, \ $R = r_1 = r_{x_1y} = 0.821$, and the multiple correlation R is greater than the individual correlation r_{x_2y} that is $R > r_2 = r_{x_2y}$ $(0.821 > 0.720)$, but less than the individual correlation $r_{x_1x_2}$ that is $R < r_3 = r_{x_1x_2}(0.821 < 0.868)$. Next, the values of the correlation coefficients between two variables and the multiple correlation coefficient will be categorized based on the interpretation in Table 6 below.

Table 6 Interpretation of Correlation Coefficient

Coefficient Interval	Level of Correlation
0.00 - 0.199	Very Low
0.20 - 0399	Low
0.40 - 0.599	Medium
0,60 - 0.799	Strong
0.80 - 1.000	Very Strong

(Sugiyono, 2017: 231)

Based on the interpretation of the correlation coefficient values in Table 6, the values $r_1 = r_{x_1y} = 0.821$, $r_2 = r_{x_2y} = 0.720$, $r_3 = r_{x_1x_2} = 0.868$, and $R = 0.821$ can be interpreted as having a very strong relationship. Meanwhile, the value of the coefficient r_2 can be interpreted as having a strong relationship only.

The conclusion is that because the samples were not randomly selected, the multiple correlation coefficients found in this study cannot be generalized to the population from which the samples were drawn. This study explains the strength of the relationship between variable x_2 and variable y, and the very strong relationship between variables x_1y, x_1x_2, and the multiple correlation x_1x_2y. Thus, the results of this study indicate that the ability to recognize proof techniques and the ability to recognize definitions/theorems have a very strong relationship with proof-solving ability, meaning that both of these abilities are highly needed in the process of proof.

Based on the findings above, there is a very strong relationship between the ability to recognize proof techniques and the ability to prove. This is consistent with the research by (Minggi et al., 2021) which suggests that the process of proof involves learning paths applied sequentially, including proof by simple procedures, existential proof involving existential proof and non-existential facts or concepts, proof by complex thinking procedures, and proof by construction. There are two types of proof techniques in mathematics, direct and indirect proofs (Celluci, 2008) (Hsu & Jos, 2008) (Stefanowicz et al., 2014).

A direct proof assumes certain hypotheses or other known statements and then logically concludes a conclusion. An indirect proof, also known as proof by contradiction, assumes the hypothesis (if given) along with the negation of a conclusion to reach a contradictory statement.

This is often equivalent to proof by contrapositive, although slightly different (see examples). Direct and indirect proofs may also involve additional tools to reach the required conclusion, such as proof by cases or mathematical induction (Stefanowicz et al., 2014) (Weber, 2004) (A. J. Stylianides, 2007). Proof by counterexample is an example of such a proof scheme (Kanellos et al., 2018).

Schemes like these can lead students to recognize proof techniques. Proof techniques are crucial knowledge for initiating proof (Stefaneas & Vandoulakis, 2015) (Hanna, 2014) and can determine the validity of arguments (Feriyanto, 2018). In the proof process, constructing evidence (Atwood, 2001) (Alcock & Weber, 2010) (Hamdani et al., 2023) (A. J. Stylianides & Stylianides, 2009) (Selden et al., 2018) (Selden et al., 2016) and proof techniques (Hsu & Jos, 2008) are required. Proof techniques and evidence construction are part of the proof framework. In writing a proof framework, it helps students to start proving (A. Selden et al., 2018) (A. Selden et al., 2016).

In addition to creating a framework by following appropriate proof techniques, it is also necessary to memorize definitions, theorems, or their proofs when composing proofs (Laamena et al., 2018) (Richard et al., 2016) (A. Selden & Selden, 2016) (J. Selden et al., 2014). Understanding and being able to work from each idea in the definition/principle (theorem) in every proof demonstrates how the correct definitions emerge from a process where patterns are observed, theorems and their proofs are found, and then counterexamples are generated until reassessment of underlying assumptions and definitions is done (Bressoud, 2020) (Ashari & Salwah, 2021) and connected with other basic mathematical concepts (Patterson, 2020). Basic concepts in school mathematics have a significant relationship with university mathematics (Junati et al., 2023). The strength of the relationship between understanding and the ability to work in using previous definitions/theorems/concepts in the proof process greatly determines success in proving abilities.

Understanding and using definitions/theorems is an important part of problem-solving and engaging in analytical thinking processes, so the analysis will be developed according to the given problem (Helma et al., 2018). In the learning path, it starts with providing experiences of understanding proofs through careful definitions, the ability to write the negation of given definitions, and being able to explain definitions even through visual illustrations (Minggi et al., 2021). Additionally, reasoning skills in the process of recognizing definitions/theorems are equally important (Tall, 2014). The presentation sequence of proofs must indeed consider the sequence of concepts, definitions, and previous theorems (Dawson, 2006) (Minggi et al., 2021) (Morali & Filiz, 2023). Definitions and the structure of theorems or axioms are connected to the steps of proving as part of proof techniques (Weber, 2002). Previously established theorems can be used to deduce new theorems or may refer to axioms, which serve as starting points as universally accepted rules (Stefanowicz et al., 2014). Mathematical proofs are absolute, meaning once a theorem is proved, it is proven forever and accepted as truth (Stefanowicz et al., 2014).

Understanding the requirements of each proof is crucial to consider (CadwalladerOlsker, 2011) (Ersen, 2016) (Güler, 2016) (Herizal et al., 2019) (G. J. Stylianides & Stylianides, 2017) (Panse et al., 2018), as all forms of proof must be proven true (valid) and follow logical (rational) deductive rules, requiring evaluation (Inglis et al., 2013) to ensure their correctness (Rocha, 2019). The truthfulness in every proof, whether done in the process of theorem simplification, can be used to determine the proof plan (Khusna, 2020), including defining proofs as logical deductions, which can be used to verify, explain, systematize, discover, and communicate mathematics as a written argument and proof action (Rocha, 2019). To enhance proof skills, conceptualization skills are required as cognitive resource-based skills (Sommerhoff et al., 2021).

Conclusion

This study investigated the correlation between proof-solving ability, the ability to recognize proof techniques, and the skill of recognizing definitions/theorems among first-year mathematics education students in Indonesia. The results revealed strong correlations among these variables.

The mean score for the ability to recognize proof techniques (x_1) was 65.90, indicating that students are fairly capable in this area. However, the mean score for the ability to recognize definitions/theorems (x_2) was lower at 60.37, suggesting potential difficulty in linking definitions/theorems to proofs. On the other hand, the mean score for proof-solving ability (y) was 68.27, indicating that, on average, students perform well in proof-solving.

The correlation coefficient between the ability to recognize proof techniques (x_1) and proof-solving ability (y) was 0.821, indicating a very strong relationship. Similarly, the correlation coefficient between the ability to recognize definitions/theorems (x_2) and proof-solving ability (y) was 0.720, indicating a strong relationship. Moreover, the correlation coefficient between the ability to recognize proof techniques (x_1) and the ability to recognize definitions/theorems (x_2) was 0.868, showing a very strong relationship.

The multiple correlation coefficient between the ability to recognize proof techniques (x_1) and the ability to recognize definitions/theorems (x_2) with proof-solving ability (y) was 0.821, indicating a very strong overall relationship. These findings suggest that both the ability to recognize proof techniques and the ability to recognize definitions/theorems are highly correlated with proof-solving ability.

The study underscores the importance of understanding proof techniques and definitions/theorems in improving proof-solving skills. Direct and indirect proof techniques play crucial roles in the proof process, and understanding definitions and theorems is essential for constructing valid proofs. Additionally, the study highlights the significance of following a systematic framework in proof-writing, starting with clear definitions and principles.

The results of this study suggest that students' ability to recognize proof techniques and definitions/theorems significantly contributes to their proof-solving ability. These findings have implications for mathematics education, emphasizing the need for structured learning approaches that focus on developing these key skills. Further research is warranted to explore interventions that can enhance students' proficiency in proof-solving.

Authorship Contribution Statement

The researchers extend their sincere gratitude to the Research and Community Service Institute of IKIP PGRI Bojonegoro for granting permission and generously funding the research conducted by the researchers in mathematics education program, Faculty of Mathematics and Sciences Education (FPMIPA), IKIP PGRI Bojonegoro. This support has been invaluable in facilitating this study and advancing research in this field.

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